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Weak Observables in MV Algebras

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A notion of a weak observable is defined and a construction of a weak observable is examined. With the help of the construction, the sum of weak observables is realized as well as the upper and lower limits of a sequence of weak observables.

1. INTRODUCTION

An MV algebra (Chang, 1958; Chovanec, 1993; Jakubík, 1995; Jurečková, 1995; Mesiar, 1994; Mundici, 1986; Riečan, n.d.-a, b); Vrábel, 1995; Vrábelová, 1995) is an algebraic system $(M, \oplus, \odot, *, 0, 1)$, where $\oplus$ and $\odot$ are binary operations, * is a unary operation, and 0, 1 are fixed elements, and some identities are satisfied.

By the Mundici representation theorem (Mundici, 1986) there is a commutative lattice ordered group G and a strong unit u in G such that

$$M = \langle 0, u \rangle, \qquad a \oplus b = (a + b) \wedge u$$

$$a \odot b = (a + b - u) \vee 0, \qquad a^* = u - a$$

and u plays the role 1 in M, and the neutral element 0 of G plays the role 0 in M. We may work with elements of an MV algebra as elements of a commutative group. E.g., if $a \leq b,\ a,\ b \in M$, then there is in G the element $b - a$ (i.e., such an element that $a + (b - a) = b$) and, because of $a \leq b$, the element $b - a \in \langle 0, u \rangle = M$. Similarly in the following definition of an observable, if $A \cap B = \emptyset$, then $x(A) + x(B) = x(A \cup B) \in \langle 0, u \rangle$, hence $x(A) \oplus x(B) = (x(A) + x(B)) \wedge u = x(A) + x(B)$. So, we can work with the group operations + instead of the MV algebra operation $\oplus$.

A state (Chovanec, 1993) is a mapping $m\colon M \to \langle 0, 1 \rangle$ such that $m(1) = 1$, $m(a_n) \nearrow m(a)$, whenever $a_n \nearrow a$ and $m(a) = m(b) + m(c)$, whenever

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$a = b \oplus c$, $b \le c^*$. An observable is a mapping x: $\mathcal{B}(R) \to M$ such that $x(R) = 1$, $x(A_n) \nearrow x(A)$ whenever $A_n \nearrow A$ and $x(A \cup B) = x(A) + x(B)$ whenever $A \cap B = \emptyset$.

In this article we shall consider weak observables instead of observables replacing the condition $x(R) = 1$ by a weaker condition $m(x(R)) = 1$.

In Section 2 we construct a weak observable from a function F: $R \to M$. Of course, we need a special property of M, so-called weak σ-distributivity (Fremlin, 1975; Wright, 1971). Recall that in the case that M is a vector lattice, the weak σ-distributivity is a necessary and sufficient condition for the extendability of any M-valued measure from a ring to the generated σ-ring.

Using these results, we construct a sum of two (independent) weak observables. We prove the commutative and the associative law.

2. CONSTRUCTION OF A WEAK OBSERVABLE

We shall assume that M is an MV-algebra satisfying the following two conditions:

(i) M is σ-complete, i.e., every sequence of elements in M has the supremum and the infimum.

(ii) M is weakly σ-distributive, i.e., for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ ($j \to, \infty$, $i = 1, 2, \ldots$) there is

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0$$

By a weak observable (with respect to a state m) we mean a mapping x: $\mathcal{B}(R) \to M$ satisfying the following properties:

(i) $m(x(R)) = 1$.

(ii) If $A, B \in \mathcal{B}(R)$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \in \mathcal{B}(R)$ ($n = 1, 2, \ldots$), $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Theorem 1. Let F: $R \to M$ be a mapping satisfying the following conditions:

(i) If $t_1 < t_2$, then $F(t_1) \le F(t_2)$.

(ii) If $t_n \nearrow t$, then $F(t_n) \nearrow F(t)$.

(iii) $\lim_{t\to\infty} m(F(t)) = 1$.

(iv) $\lim_{t\to-\infty} m(F(t)) = 0$.

Then there exists a weak observable x: $\mathcal{B}(R) \to M$ such that $m(x((-\infty, t))) = m(F(t))$ for every $t \in R$.

Proof. We shall use two results from Riečan (n.d.-a) concerning M-valued measures. An M-valued measure is a mapping μ: $\mathcal{R} \to M$ such that $\mu(\emptyset) = 0$, μ is additive and continuous. By a corollary of the Alexandrov theorem (Theorem 2 in Riečan, n.d.-a), from conditions (i) and (ii) the existence follows of a measure $\lambda_F: \mathcal{R} \to M$ defined on the ring $\mathcal{R}$ generated by all intervals of the form $\langle a, b)$ such that

$$\lambda_F(\langle a, b)) = F(b) - F(a)$$

for every $a, b \in \mathcal{R}$, $a < b$. Now the measure extension theorem (Riečan, n.d.-a, Theorem 3) is applicable: There exists a measure (denote it by x) defined on the σ-algebra $\sigma(\mathcal{R}) = \mathcal{B}(R)$ with values in M and extending λ_F. We shall prove that x is a weak observable. Indeed, $x: \mathcal{B}(R) \to M$ is additive and continuous.

Moreover,

$$\begin{aligned} m(x(R)) &= m\left(x\left(\bigcup_{n=1}^{\infty} \langle -n, n)\right)\right) = m\left(\bigvee_{n=1}^{\infty} x(\langle -n, n))\right) \\ &= m\left(\bigvee_{n=1}^{\infty} \lambda_F(\langle -n, n))\right) = m\left(\bigvee_{n=1}^{\infty} (F(n) - F(-n))\right) \\ &= \lim_{n\to\infty} m(F(n)) - \lim_{n\to\infty} F(-n) = 1 - 0 = 1 \end{aligned}$$

Finally,

$$\begin{aligned} m(x((-\infty, t))) &= m\left(\left(\bigcup_{n=1}^{\infty} \langle t - n, t\right)\right) = m\left(\bigcup_{n=1}^{\infty} F(\langle t - n, t))\right) \\ &= m(F(t)) - \lim_{n\to\infty} m(F(t - n)) = m(F(t)) \end{aligned}$$

3. SUM OF WEAK OBSERVABLES

Of course, the classical case seems to be the most illustrative. If $(\Omega, \mathcal{S}, P)$ is a probability space, then every random variable $\xi: \Omega \to R$,) is an $\mathcal{S}$-measurable function. Putting $x(A) = \xi^{-1}(A)$, we obtain an observable x: $\mathcal{B}(R) \to \mathcal{S}$, $\mathcal{S}$ being an MV algebra with $A \oplus B = A \cup B$ and $A \odot B = A \cap B$. Hence in the case of the sum of observables we can be inspired by the sum $\xi + \eta$ of random variables ξ, η:

$$\begin{aligned} &(\xi + \eta)^{-1}((-\infty, t)) \\ &= \bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n} \xi^{-1}\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cap \eta^{-1}\left(\left\langle \frac{j-1}{2^n}, \frac{j}{2^n}\right)\right) \end{aligned}$$

where $(i,j) \in \alpha_n$ if and only if $i/2^n + i/2^n < t$. In the MV algebra M-valued case we shall substitute $\cup_{n=1}^{\infty}$ by $\vee_{n=1}^{\infty}$, $\cup_{(i,j)} \in \alpha_n$ by $\Sigma_{(i,j)\in\alpha_n}$ and $\cap$ by the operation $\odot$.

We have restricted our considerations to the case of the operation $\odot$. The second restriction will concern the case of $\odot$-independent observables. Two weak observables x,y are called $\odot$-independent if

$$m(x(A) \odot y(B)) = m(x(A)) \cdot m(y(B))$$

for every $A, B \in \mathcal{B}(R)$. More generally, weak observables $x_1, \ldots, x_n$ are $\odot$-independent if

$$m(x_1(A_1) \odot x_2(A_2) \odot \cdots \odot x_n(A_n)) = m(x_1(A_1)) \cdot m(x_1(A_1)) \cdot \ldots \cdot m(x_1(A_n))$$

for every $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$.

If $m: M \to \langle 0, 1\rangle$ is a state and $x: \mathcal{B}(R) \to M$ is a weak observable, then the composite mapping $m_x = m \circ x: \mathcal{B}(R) \to \langle 0, 1\rangle$ defined by the formula $m_x(A) = m(x(A))$ is a probability measure. Evidently

$$m(x(A)) \cdot m(y(B)) = m_x \times m_y(A \times B)$$

where $m_x \times m_y$ is the product of the measures m_x, m_y. Therefore $\odot$ independence of x, y can be formulated by the equality

$$m(x(A) \odot y(B)) = m_x \times m_y(A \times B), \qquad A, B \in \mathcal{B}(R)$$

Put $A_i^n = \langle (i-1)/2^n, i/2^n)$, $i \in Z$, $n \in N$, $\alpha_n(t) = \{(i, j); i + j < 2^n t\}$. We want to work with the sums

$$\gamma_n(t) = \sum_{(i,j)\in\alpha_n(t)} x(A_i^n) \odot y(A_j^n)$$

Of course, since the distributive law for the operations $\oplus$ and $\odot$ need not hold, we are not able to prove that $\gamma_n(t) \leq \gamma_{n+1}(t)$. Therefore instead of $\gamma_n(t)$ we define first

$$\beta_1(t) = \alpha_1(t)$$

$$\beta_n(t) = \alpha_n(t) \setminus \{(i, j); \exists m < n, \exists (k, l) \in \alpha_m(t), A_i^n \times A_j^n \subset A_k^m \times A_l^m\}$$

$$\Gamma_n(t) = \sum_{m=1}^{n} \sum_{(i,j)\in\beta_m(t)} x(A_i^n) \odot y(A_j^n)$$

Theorem 2. Let M be a σ-complete MV algebra, $x, y: \mathcal{B}(R) \to M$ be $\odot$-independent weak observables. Define $\Gamma_n(t)$ as above and

$$F(t) = \bigvee_{n=1}^{\infty} \Gamma_n(t)$$

Then F satisfies the assumptions (i)–(iv) of Theorem 1.

Proof. First note that a semidistributive law holds:

$$(a + b) \odot c \geq (a \odot c) + (b \odot c)$$

Let $t < s$. By the semidistributive law $\Gamma_n(t) \leq \Gamma_n(s)$ $(n = 1, 2, \ldots)$, hence $F(t) \leq F(s)$.

Assume $t_k \nearrow t$. Denote $S = \sup \{i + j;\ (i, j) \in \beta_n(t)\}$. It is easy to see that S is an integer, $S < 2^n\, t$. Therefore there exists k such that $S < 2^n t_k$. Hence to every n there is k such that $\beta_n(t) \subset \beta_n(t_k)$. Denote

$$F(t) = \bigvee_n a_n, \qquad F(t_k) = \bigvee_n b_{n,k}$$

Evidently $F(t_k) \leq F(t)$, hence $\vee_k F(t_k) \leq F(t)$. On the other hand, we have proved that to every n there is k such that $\beta_n(t) \subset \beta_n(t_k)$ hence $a_n \leq b_{n,k}$. Therefore

$$a_n \leq b_{n,k} \leq \bigvee_n b_{n,k} = F(t_k) \leq \bigvee_k F(t_k)$$

$$F(t) = \bigvee_n a_n \leq \bigvee_k F(t_k)$$

We have proved (i) and (ii). For to prove (iii) and (iv), we first prove

$$m(F(t)) = m_x \times m_y(\{(u, v);\ u + v < t\}) \tag{$*$}$$

Indeed

$$m(F(t)) = \lim_{n\to\infty} \sum_{m=1}^{n} \sum_{(i,j)\in\beta_m(t)} m\left(x\left(\left\langle\frac{i-1}{2^n}, \frac{i}{2^q n}\right)\right) \odot y\left(\left\langle\frac{j-1}{2^n}, \frac{j}{2^n}\right)\right)\right)$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} \sum_{(i,j)\in\beta_m(t)} m_x \times m_y\left(\left\langle\frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left\langle\frac{j-1}{2^n} \frac{j}{2^n}\right)\right)$$

$$= m_x \times m_y\left(\bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\alpha_n(t)} \left\langle\frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left\langle\frac{j-1}{2^n}, \frac{j}{2^n}\right)\right)$$

$$= m_x \times m_y(\{(u, v);\ u + v < t\})$$

Since $m_x \times m_y$ is a probability measure, (iii) and (iv) follow by $(*)$.

By Theorem 1 there is a weak observable z such that $m(z((-\infty, t))) = m(F(t))$ for every $t \in R$. This function z will be called a sum of weak observables x, y and will be denoted by $x + y$.

Theorem 3. Let M be a σ-complete, weakly σ-distributive MV algebra. Let x, y, z: $\mathcal{B}(R) \to M$ be $\odot$-independent weak (with respect to m) observables. Then the following properties are satisfied:

(i) $m_{x+y} = m_{y+x}$ (commutative law).

(ii) $m_{(x+y)+z} = m_{x+(y+z)}$ (associative law).

Proof. By the definition of $x + y$ and the equality $(*)$ (see Proof of Theorem 2),

$$\begin{aligned} m_{x+y}((-\infty, t)) &= m((x + y)((-\infty, t))) = m(F(t)) \\ &= (m_x \times m_y)(g^{-1}((-\infty, t))) \end{aligned}$$

where $g\colon R \times R \to R$ is defined by $g(u, v) = u + v$. Therefore

$$m_{x+y} = (m_x \times m_y) \circ g^{-1} \tag{$**$}$$

By $(**)$ the commutative law follows. Further,

$$\begin{aligned} m_{(x+y)+z} &= (m_{x+y} \times m_z) \circ g^{-1} \\ &= (((m_x \times m_y) \circ g^{-1}) \times m_z) \circ g^{-1} \end{aligned}$$

hence by the Fubini theorem

$$\begin{aligned} &m_{(x+y)+z}((-\infty, t)) \\ &\quad = (m_{x+y} \times m_z)(\{(u, v); u + v < t\}) \\ &\quad = \int_R m_{x+y}((-\infty, t - u))\, dm_z(u) \\ &\quad = \int_R m_x \times m_y(\{(w, v); v + w < t - u\})\, dm_z(u) \\ &\quad = \int_R \left(\int_R m_x((-\infty, t - u - v))\, dm_z(v) \right) dm_z(u) \\ &\quad = m_x \times m_y \times m_z(\{(w, v, u); u + v + w < t\}) \end{aligned}$$

By the equality

$$m_{(x+y)+z}((-\infty, t)) = m_x \times m_y \times m_z(\{(u, v, w); u + v + w < t\})$$

the equality

$$m_{(x+y)+z}((-\infty, t)) = m_{x+(y+z)}((-\infty, t))$$

follows, hence

$$m_{(x+y)+z} = m_{x+(y+z)}$$

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